CALCULATING THE PARAMETERS OF AN ELECTRON BEAM THAT DRIFTS ACROSS A STRONG HOMOGENEOUS MAGNETIC FIELD

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Electron drift in specified fields has been examined in [1] and, as applied to a magnetron, in [2–4] with the averaging method. In [1, 2], a first- and in [3, 4] in a second-order approximation of the small parameter $\nu \sim \eta E/\Omega^2 L$ was used. Here and below, E and $H \equiv (c/\eta)\Omega$ are the field strengths, L is the characteristic dimension of the field heterogeneity, η is the charge-mass ratio of an electron ($\eta > 0$), and c is the velocity of light. An attempt to construct similar approximations for a drifting electron beam with allowance for the space-charge field, within the framework of the averaging method, involves considerable mathematical difficulties. This paper describes an attempt to solve the latter problem for a stationary monoenergetic beam that drifts under the influence of a plane electric field with potential $\varphi(x, y)$ across a strong homogeneous magnetic field $H_z \equiv H = \text{const}$. Solutions are constructed by the method of successive approximations, in powers of the parameter $\varepsilon = h/L$, where h is the Larmor electron radius for narrow beams with a width on the order of 2h.

§1. Single-flow beam. Let the curve $x = x_c(l)$, $y = y_c(l)$—the beam axis—be situated near the central trajectory of a plane beam. The system of orthogonal coordinates s, l, in which l = const (let the right orthogonal axes s = 0) is associated with the beam and is determined by the curvature of the axis [5].

$$x = x_c + sy'_c;\ \ y = y_c - sx_c';\ \ x_c' \equiv dx_c/dl;$$

$$dx^2 + dy^2 = ds^2 + g\,dl^2 \quad g \equiv (1 - ks)^2;$$

$$k \equiv y_c' x_c'' - x_c' y_c''. \tag{1.1}$$

1.1°. In the coordinates s, l, the equations of a nonrelativistic monoenergetic beam with space-charge density $\rho(x, y)$ have the form

$$\varepsilon_*^2 v_s^2 + \frac{1}{g}\Gamma_l^2 = 2\eta\varphi;\quad \frac{\partial v_l}{\partial s} - \varepsilon_*^2\frac{\partial v_s}{\partial l} = \Omega\sqrt{g};$$

$$\frac{\partial}{\partial s}\sqrt{g}\frac{\partial\varphi}{\partial s} + \varepsilon_*^2\frac{\partial}{\partial l}\frac{1}{\sqrt{g}}\frac{\partial\varphi}{\partial l} = 4\pi\rho\sqrt{g}\,\cdot$$

$$\frac{\partial}{\partial s}\sqrt{g}\rho v_s + \frac{\partial}{\partial l}\frac{1}{\sqrt{g}}\rho v_l = 0;$$

$$g \equiv (1 - \varepsilon_* ks)^2;\quad \Omega \equiv \frac{\eta}{c}H. \tag{1.2}$$

In Eqs. (1.2) we place the small parameter ε_* at those places where $\varepsilon = h/L$ appears as a result of conversion to the dimensionless values

$$\frac{v_s}{\varepsilon\Omega h};\ \ \frac{v_l}{\Omega h};\ \ U \equiv \frac{\eta\varphi}{\Omega^2 h^2},\ \ q \equiv \frac{s}{h};$$

$$\lambda \equiv \frac{l}{L};\ \ n \equiv \frac{4\pi\eta}{\Omega^2}\rho \quad \varkappa \equiv kL. \tag{1.3}$$

We should add to Eqs. (1.2) the obvious condition of constancy of the total current J_c of a free beam,

$$\int_{s_-}^{s_+}\rho v_l\frac{ds}{\sqrt{g}} = J_c \equiv \frac{\Omega^3 h^2}{4\pi\eta}i_c = \text{const}. \tag{1.4}$$

1.2°. The solution of system (1.2), for a narrow beam $q \ll 1$ of small curvature $\varkappa \ll 1$, can be sought as a series in powers of ε. After simple computations with accuracy to ε^3, we obtain the following (dimensionless) result:

$$n = 1 + 2\varepsilon\varkappa(q + \gamma) +$$

$$+ \varepsilon^2[\varkappa^2(5q^2 + 8\gamma q + 2\gamma^2) - 2q\gamma^{\cdot\cdot} + 2\gamma^{\cdot 2}];$$

$$U = (q + \gamma)^2 + \varepsilon\varkappa(q^3 + 3\gamma q^2 + 2\gamma^2 q) +$$

$$+\varepsilon^2(^5/_4\varkappa^2 q^4 + 4\varkappa^2\gamma q^3 + 3\varkappa^2 q^2\gamma^2 -$$

$$- q^2(q + \gamma)\gamma^{\cdot\cdot} + q^2\gamma^{\cdot 2});\qquad \gamma^{\cdot} \equiv d\gamma/d\lambda;$$

$$i \equiv \frac{4\pi\eta}{\Omega^3 h^3}J = \int_{q_-}^{q} nV_l\frac{dg}{g} =$$

$$= [\gamma q + {}^1/_2 q^2 + \varepsilon\varkappa(^5/_6 q^2 + {}^5/_2\gamma q^2 + 2\gamma^2 q) +$$

$$+ \varepsilon^2\varkappa^2(^{13}/_8 q^4 + {}^{17}/_3\gamma q^3 + 6\gamma^2 q^2 +$$

$$+ 2\gamma^3 q) + \varepsilon^2(\gamma^{\cdot 2}q^2 + 2\gamma^{\cdot 2}\gamma q - {}^5/_6\gamma^{\cdot\cdot}q^3 - \gamma\gamma^{\cdot\cdot}q^2)]_{q_-}{}^{q};$$

$$V_s \equiv \frac{v_s}{\varepsilon\Omega h} = -\frac{1}{gn}\frac{\partial i}{\partial\lambda} = -\gamma^{\cdot}q + \ldots;\ \ V_l \equiv \frac{v_l}{\Omega h}. \tag{1.5}$$

Here, $\gamma(\lambda)$—an arbitrary function—is related to the beam boundaries by condition (1.4). If we take as the axis the central trajectory $q_+ = -q_- = b$, condition (1.4), in view of (1.5), takes the form

$$\gamma b + \varepsilon\varkappa(^5/_6 b^3 + 2\gamma^2 b) + \varepsilon^2\varkappa^2(^{17}/_3\gamma b^3 + 2\gamma^3 b) +$$

$$+ \varepsilon^2(2\gamma^{\cdot 2}\gamma b - {}^5/_6\gamma^{\cdot\cdot}b^3) = {}^1/_2 i_c = \text{const}. \tag{1.6}$$

Results (1.5) overlaps in many ways with the results of [6] and, generally speaking, could have been obtained by expansion of the solution in powers of s, which was examined in [6]. However, the remaining results of this paper require a more general approach. If we represent the external harmonic field $\varphi_\pm$ near the beam as

$$\frac{\eta\varphi_\pm}{\Omega^2 h^2} = V_\pm(l) + qB_\pm(l) + \frac{\varepsilon}{2}\varkappa q^2 B_\pm +$$

$$+ \frac{\varepsilon^2}{6}(2\varkappa^2 B_\pm - B_\pm^{\cdot\cdot})q^3 - \frac{\varepsilon^2}{2}V_\pm q^2,$$

in view of (1.5), when the axis $q_\pm = \pm$' is centrally positioned, it is easy to obtain the relations

$$2V_\pm \pm 2B_\pm b - (\gamma \pm b)^2 + \varepsilon\varkappa(B_\pm b^2 \mp b^3 - 3\gamma b^2 \mp 2\gamma^2 b) +$$

$$+ \varepsilon^2\varkappa^2(\mp{}^2/_3 B_\pm b^3 - {}^5/_4 b^4 \mp 4\gamma b^3 - 3\gamma^2 b^2) -$$

$$- \varepsilon^2(V^{\cdot\cdot}_\pm q^2 \pm {}^1/_3 B^{\cdot\cdot}_\pm b^3 + \gamma^{\cdot 2}b^2 - \gamma^{\cdot\cdot}\gamma b^2 \mp \gamma^{\cdot\cdot}b^3), \tag{1.7}$$

$$B_\pm = (\gamma \pm b) - \varepsilon\varkappa(\pm B_\pm b - {}^3/_2 b^2 \mp 3\gamma b - \gamma^2) -$$

$$- \varepsilon^2\varkappa^2(B_\pm b^2 \mp {}^5/_2 b^3 - 6\gamma b^2 \mp 3\gamma^2 b) +$$

$$+ \varepsilon^2(^1/_2 b^2 B^{\cdot\cdot}_\pm \pm V_\pm b \pm \gamma^{\cdot 2}b - {}^3/_2\gamma^{\cdot\cdot}b^2 \mp \gamma^{\cdot\cdot}\gamma b), \tag{1.8}$$

which result in the expression for the field $\varphi_\pm$ in the inverse problem, when the axis is given. If, for example, we consider a beam in a narrow channel with specified wall potentials (direct problem), then (1.7) leads to a first-order differential equation (in zeroth approximation) for the axis.

1.3°. If expansion of the metric function g in ε_* is abandoned, the solution of (1.2) can be sought as series in powers of ε_*^2. With accuracy to ε^4, it follows from (1.2) that

$$2\eta\varphi = \varepsilon_*^2 v_{s0}^2 + \frac{1}{g} v_l^2; \quad v_l = \frac{\Omega}{2k}(\Gamma - g) + \varepsilon_*^2 \frac{\partial}{\partial q}\int_0^s v_{s0} ds;$$

$$v_{s0} = \frac{\Omega}{4k^2\sqrt{g\rho_0}} \frac{\partial}{\partial l}\int_1^g \rho_0 (\Gamma - g)\frac{dg}{g} \equiv \frac{1}{\sqrt{g\rho_0}} \frac{\partial}{\partial l} J_0;$$

$$4\pi\rho = 4k^2 \frac{\partial}{\partial g} g \frac{\partial\varphi}{\partial g} + \frac{\varepsilon_*^2}{\sqrt{g}} \frac{\partial}{\partial l}\left(\frac{1}{\sqrt{g}}\frac{\partial\varphi_0}{\partial l}\right)_s; \quad g \equiv (1 - ks)^2. \quad (1.9)$$

Here, Γ is an arbitrary function of l; the beam axis was taken for convenience at the lower boundary $s_- = 0$ (see Fig. 1), the derivatives with respect to l are taken at fixed s; the zero subscript indicates zeroth approximation in ε^2;

$$2\eta\varphi_0 = \frac{\Omega^2}{4k^2 g}(\Gamma - g)^2; \qquad \eta\frac{\partial\varphi_0}{\partial g} = \frac{\Omega^2}{4k^2}\left(1 - \frac{\Gamma^2}{g^2}\right);$$

$$\frac{8\pi\eta}{\Omega^2}\rho_0 = 1 + \frac{\Gamma^2}{g^2}; \qquad V_{l0} = \frac{\Omega}{2k}(\Gamma - g);$$

$$J_0 \frac{32\pi\eta k^2}{\Gamma\Omega^3} = \frac{g}{\Gamma} +$$

$$+ \frac{\Gamma^2}{2g^2} - \ln\frac{g}{\Gamma} - \frac{\Gamma}{g} - \frac{1}{\Gamma} - \frac{\Gamma^2}{2} + \ln\frac{1}{\Gamma} + \Gamma. \quad (1.10)$$

As one would expect, expressions (1.10) coincide with the exact solution for a cylindrical one-dimensional (r) beam if Γ = const, k = const, and g = r^2. Thus, the old formulas have new contents here: Γ and k can be any functions of l and are related to the beam current J_c = const and width $s_+ = a(l)$ by a condition that follows from (1.4) and (1.10):

$$k_*^2 = \Gamma[-z + {}^1/_2\, z^2 + \ln z + {}^1/_z -$$

$$- {}^1/_\Gamma - {}^1/_2\,\Gamma^2 - \ln\Gamma + \Gamma];$$

$$k_* \equiv k(32\pi\eta J_c)^{1/2}\,\Gamma^{-1/2}\Omega^{-3/2}; \quad z \equiv \Gamma(1 - ka)^{-2}. \quad (1.11)$$

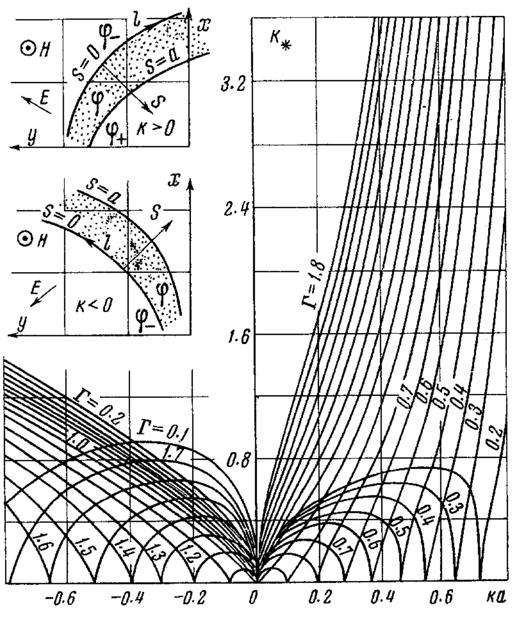

Fig. 1

The external field $\varphi_\pm$ near the beam can, using the Laplace equation, be represented with accuracy to ε^2 as

$$\varphi_\pm = \psi_\pm(l) - B_\pm(l)\ln g. \quad (1.12)$$

The conditions of field continuity give at the lower boundary s = 0

$$8\eta k^2\psi_- = \Omega^2(\Gamma - 1)^2, \quad 8\eta k^2 B_- = \Omega^2(\Gamma^2 - 1). \quad (1.13)$$

Hence follows the equation for the lower boundary

$$8\eta k^2\varphi = \Omega^2[(1 + 8\eta k\Omega^{-2} A)^{1/2} - 1]. \quad (1.14)$$

which is actually the equation of the electron trajectory, where $\varphi = \psi_-$ is the potential and A ≡ kB_- is the field-strength component that is normal to the trajectory. At the upper boundary we have

$$B_* \equiv (8\eta k^2/\Gamma\Omega^2)(B_+ - B_-) = z - {}^1/_z - \Gamma + {}^1/\Gamma$$

$$V_* \equiv (8\,\eta k^2/\Gamma\Omega^2)(\psi_+ - \psi_-) =$$

$$= z + {}^1/_z - \Gamma - 1/\Gamma + (z - {}^1/_z)\ln\Gamma/z. \quad (1.15)$$

Figures 1–3 show graphs of k_*, B_*, and V_* as functions of ka. From these graphs, along the lower boundary of the beam, we can determine the field near the beam and the position of the electrodes $S_\pm(l)$ with potentials $\psi_\pm$

$$kS_\pm = 1 - \exp[(\psi_\pm - \Psi_\pm)/2\,B_\pm]$$

thereby solving the inverse problem for a narrow band containing a narrow beam.

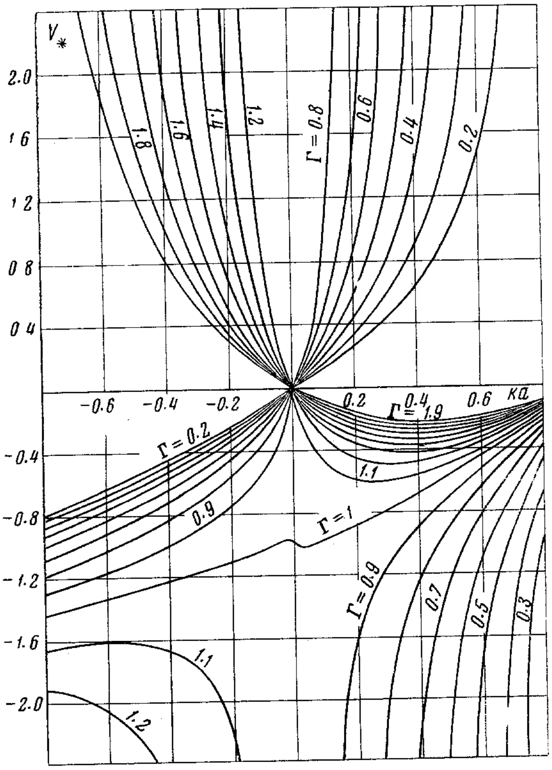

Fig. 2

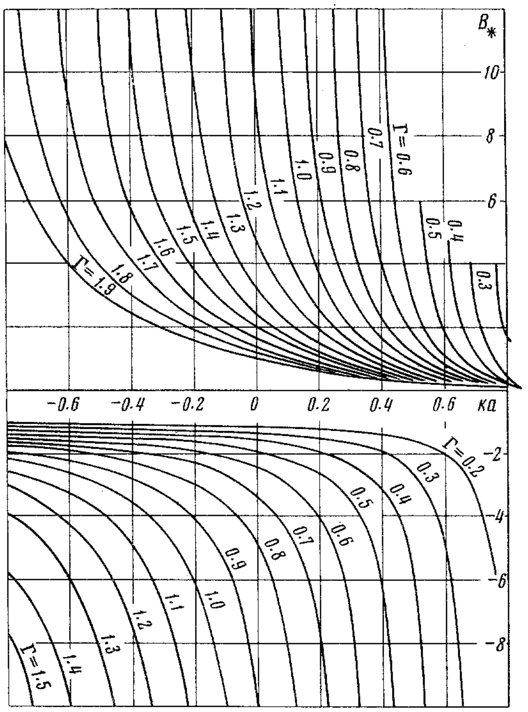

Fig. 3

1.4°. Extension of (1.2) to the relativistic case gives equations for a single-flow relativistic beam with velocity c(v/B):

$$B^2 = c^2 + \varepsilon_*^2 v_s^2 + \frac{v_l^2}{g}; \quad \frac{\partial}{\partial s} H = \frac{4\pi\eta}{c^2\sqrt{g}}\rho v_l; \tag{1.16}$$

$$\frac{\partial}{\partial s}\sqrt{g}\frac{\partial B}{\partial s} + \frac{\partial}{\partial l}\frac{\varepsilon_*^2}{\sqrt{g}}\frac{\partial B}{\partial l} = \frac{4\pi\eta}{c^2}\rho\sqrt{g}B; \quad B \equiv c + \frac{\eta}{c}\varphi; \tag{1.17}$$

$$v_s = \frac{-1}{\rho\sqrt{g}}\frac{\partial}{\partial l}\int_{s_-}^{s}\rho v_l\frac{ds}{\sqrt{g}}; \quad \frac{\eta}{c}H = \frac{1}{\sqrt{g}}\left(\frac{\partial v_l}{\partial s} - \frac{\partial v_s}{\partial l}\right). \tag{1.18}$$

With accuracy to ε^2, this system can be written as

$$\frac{4\pi\eta}{k^2c^2}\rho g = \frac{A''}{A} - 1 = \frac{B''}{B}; \quad \frac{B^2}{c^2} = 1 + A^2;$$

$$A \equiv \frac{v_l}{c\sqrt{g}}; \quad A' \equiv \frac{dA}{d\sigma}; \quad \sigma^2 \equiv \ln g. \tag{1.19}$$

The solution follows:

$$4\pi\eta(kc)^{-2}\rho g = \beta^2 + 2A^2;$$

$$\gamma + \ln\sqrt{g} = 1/\beta F\left(\mathrm{arctg}\,A,\ \sqrt{\beta - 1/\beta}\right), \tag{1.20}$$

where F is an elliptic integral of the first kind; γ and $\beta > 1$ are arbitrary functions of l, which are related to the beam width $2a(S_\pm = \pm a)$ and the current J_c by a condition that follows from (1.4), (1.16), and (1.17):

$$\frac{4\pi\eta}{kc^3}J_c = \int_{A_+}^{A_-}(\beta^2 + 2A^2)\frac{A\,dA}{\sqrt{g}\sqrt{(\beta + A^2)(1 + A^2)}};$$

$$A_\pm \equiv A|_{s=\pm a}. \tag{1.21}$$

Integral (1.21) is not expressed in elementary functions. If we expand g from (1.20) and (1.21) in powers of ε_*, with the same accuracy we obtain

$$A = \mathrm{sh}\,Q + {}^1/_2\varepsilon_*\mu ks^2\,\mathrm{ch}\,Q, \quad B = c\,(\mathrm{ch}\,Q + {}^1/_2\,\varepsilon_*\mu\,ks^2\,\mathrm{sh}\,Q);$$

$$\eta c^{-2}H = \mu\,[(1 + \varepsilon_*\,ks)\,\mathrm{ch}\,Q + {}^1/_2\,\varepsilon_*\mu\,ks^2\,\mathrm{sh}\,Q] - \varepsilon_* k\,\mathrm{sh}\,Q;$$

$$4\pi\eta\,c^{-2}\rho = \mu^2\,(1 + 2\varepsilon_* ks);$$

$$2\pi\eta c^{-3}J_c = \mathrm{sh}\gamma\ \mathrm{sh}\,\mu a + \varepsilon_* k\,\{\mu a\ \mathrm{ch}\gamma\ \mathrm{ch}\,\mu a -$$

$$- [1 + {}^1/_2(\mu a)^2]\ \mathrm{sh}\,\mu a\ \mathrm{ch}\,\gamma\}\ Q \equiv \gamma + \mu s;\ \mu \equiv k\beta.$$

The magnetic field H inside the beam is inhomogeneous. Outside of the beam, the field is homogeneous and $H_\pm = H(s = \pm a)$.

§2. Double-Flow Beam. Let us use the following representation of a two-valued velocity $v_{(1)}$, $v_{(2)}$ and density field $\rho_{(1)}$, $\rho_{(2)}$:

$$\mathbf{v}_{(1)} = \mathbf{\Gamma} + \nabla w; \ \mathbf{v}_{(2)} \equiv \mathbf{\Gamma} - \nabla w;$$

$$2\rho_{(1)} \equiv \rho + \delta; \ 2\rho_{(2)} \equiv \rho - \delta. \tag{2.1}$$

In this representation, the electron hydrodynamic equations for a nonrelativistic stationary monoenergetic beam with an irrotational field of generalized momentum have the form

$$\nabla w\mathbf{\Gamma} = 0; \ (\nabla w)^2 = 2\eta\Phi; \ 2\eta\Phi \equiv 2\eta\varphi - \Gamma^2; \tag{2.2}$$

$$\nabla(\rho\mathbf{\Gamma}) + \nabla(\delta\nabla w) = 0; \ \nabla(\delta\mathbf{\Gamma}) + \nabla(\rho\nabla w) = 0; \tag{2.3}$$

$$\nabla^2\varphi = 4\pi\rho; \ \partial\Gamma_y/\partial x - \partial\Gamma_x/\partial y = \Omega; \ \Omega \equiv \eta/c\ H. \tag{2.4}$$

By adding and subtracting Eqs. (2.2) and (2.3), it is easy to see, in view of (2.1), that (2.2) are equivalent to the energy integrals and (2.3) are equivalent to the continuity equations for the first $v_{(1)}$, $\rho_{(1)}$ and second $v_{(2)}$, $\rho_{(2)}$ subflows that form the beam in question. We isolate the currents

$$\mathbf{J} = \rho\mathbf{\Gamma} + \delta\nabla w, \ \mathbf{J}_\pm = \pm\,{}^1/_2\,(\delta\mathbf{\Gamma} + \rho\nabla w),$$

where J is the current density of the beam and $J_\pm$ is the density of the so-called rotary current. It is apparent from Eqs. (2.2) that velocity Δw (oscillatory), by which the subflows are distinguished, is orthogonal to the total (downwash) velocity Γ for both subflows and vanishes ($\nabla w = 0$) at the surfaces $\Phi = 0$. The surface $\Phi = 0$, as the boundary of a two-velocity motion, is the boundary of the beam in question,* and we may write the conditions

$$\text{(a)}\ w\,|_{\Phi=0} = \text{const and (b)}\ \mathbf{J}\nabla\Phi|_{\Phi=0} = 0, \tag{2.5}$$

where (a) is equivalent to $\nabla w = \Phi = 0$ and (b) asserts that the beam current does not intersect the beam boundaries** in the absence of sources at the boundaries. Conditions (2.5) are necessarv for the method of successive approximations.

2.1°. Let, in coordinates s, l (which are associated with the beam), the axis s = 0 be situated near the center of the beam; $s_+(l) > 0$ and $s_-(l) < 0$ explicitly express the unknown beam boundaries $\Phi = 0$. Taking (1.1) into account, Eqs. (2.2)-(2.4) are written in coordinates

*A case in which several surfaces $\Phi = 0$ are formed in the beam is possible. For simplicity, we consider the case of one wave $\Phi \geq 0$ within the beam.

**The latter is valid, generally speaking, if $(\nabla\Phi)^2 \neq \Phi = 0$. In the opposite case $(\nabla\Phi)^2 = \Phi = 0$, which is possible with a full space charge, the current can intersect the boundary $\Phi = 0$.

s, l, as

$$\frac{\partial w}{\partial s}\Gamma_s+\frac{1}{g}\frac{\partial w}{\partial l}\Gamma_l=0;$$

$$\left(\frac{\partial w}{\partial s}\right)^2+\frac{\varepsilon_*^2}{g}\left(\frac{\partial w}{\partial l}\right)^2=2\eta\Phi, \tag{2.6}$$

$$2\eta\Phi=2\eta\varphi-\varepsilon_*^2\Gamma_s^2-1/g\Gamma_l^2;$$

$$\partial\Gamma_l/\partial s-\varepsilon_*^2\partial\Gamma_l/\partial l=\Omega\sqrt{g}; \tag{2.7}$$

$$\sqrt{g}\left(\rho\Gamma_s+\delta\frac{\partial w}{\partial s}\right)=$$
$$=-\frac{\partial}{\partial l}\int\limits_{s_-}^{s}\left(\rho\Gamma_l+\varepsilon_*^2\delta\frac{\partial w}{\partial l}\right)\frac{ds}{\sqrt{g}}+\Sigma(l), \tag{2.8}$$

$$\sqrt{g}\left(\varepsilon_*^2\delta\Gamma_s+\rho\frac{\partial w}{\partial s}\right)=$$
$$=-\frac{\partial}{\partial l}\int\limits_{s_-}^{s}\varepsilon_*^2\left(\delta\Gamma_l+\rho\frac{\partial w}{\partial l}\right)\frac{ds}{\sqrt{g}}+I(l), \tag{2.9}$$

$$\frac{\partial}{\partial s}\sqrt{g}\frac{d\varphi}{ds}+\varepsilon_*^2\frac{\partial}{\partial l}\frac{1}{\sqrt{g}}\frac{\partial\varphi}{\partial l}=4\pi\delta\sqrt{g};$$

$$g\equiv(1-\varepsilon_*ks)^2;\quad \Gamma_s\equiv\Gamma_x y_c'-\Gamma_y x_c';$$

$$\Gamma_l\equiv\sqrt{g}(\Gamma_x x_c'+\Gamma_y y_c'). \tag{2.10}$$

Here, Γ_s and Γ_l are covariant components of the vector Γ in the s, l system. In the continuity equations, integration is with respect to s, so that I and Σ are arbitrary functions of l. In (2.6)–(2.10), the parameter ε_* is placed where $\varepsilon \equiv h/L$ appears as a result of conversion to dimensionless values (1.3) and

$$V_s\equiv\frac{\Gamma_s}{\varepsilon\Omega h};\quad V_l\equiv\frac{\Gamma_l}{\Omega h};\quad \vartheta\equiv\frac{w}{\Omega h^2};$$

$$u_s\equiv\frac{\partial w}{\partial s}\frac{1}{\Omega h};\quad u_l\equiv\frac{\partial w}{\partial l}\frac{1}{\Omega h}. \tag{2.11}$$

Conditions (2.5) can be written in the s, l system as

$$\Phi(s=s_+)=0;\quad \Phi(s=s_-)=0;\quad w(s=s_-)=0;$$

$$w(s=s_+)={}^1/_2\pi\Omega h^2, \tag{2.12}$$

and

$$\left[\left(\rho\Gamma_s+\delta\frac{\partial w}{\partial s}\right)\frac{\partial\Phi}{\partial s}+\right.$$
$$\left.+\frac{1}{g}\left(\rho\Gamma_l+\varepsilon_*^2\delta\frac{\partial w}{\partial l}\right)\frac{\partial\Phi}{\partial l}\right]_{s=s_+}=0. \tag{2.13}$$

In view of (2.8) and the identities

$$\left[\frac{\partial\Phi}{\partial l}+\frac{ds_\pm}{dl}\frac{\partial\Phi}{\partial s}\right]_{s=s_\pm}=0, \tag{2.14}$$

conditions follow from (2.13) that determine Σ and I:

$$\Sigma=0;\quad \int\limits_{s_-}^{s_+}\left(\rho\Gamma_l+\varepsilon_*^2\delta\frac{\partial w}{\partial l}\right)\frac{ds}{\sqrt{g}}=J_c\equiv\frac{\Omega^3h^2}{4\pi\eta}i_c, \tag{2.15}$$

where J_c is the total beam current.

2.2°. With accuracy to ε^2, we can rewrite (2.6)–(2.10), (2.12), and (2.15) in dimensionless variables (1.3) and (2.11) as

$$2U^2=u_s^2+(\gamma+q)^2+\varepsilon\varkappa(2\gamma^2q+3\gamma q^2+q^3); \tag{2.16}$$

$$V_l=\gamma+q-{}^1/_2\varepsilon\varkappa q^2;\quad V_s=-(\gamma+q);\ (v^{\cdot}/u_s) \tag{2.17}$$

$$v^{\cdot}\equiv dv/d\lambda;$$

$$n(1-\varepsilon\varkappa q)=\frac{\alpha}{u_s};\quad \alpha\equiv\frac{4\pi\eta I}{\Omega^3h};$$

$$\frac{\partial}{\partial q}(1-\varepsilon\varkappa q)\frac{\partial U}{\partial q}=\frac{\alpha}{u_s}; \tag{2.18}$$

$$u_s|_{q=q_\pm}=0;\quad \int\limits_{q_-}^{q_+}u_s dq=\frac{\pi}{2};$$

$$\int\limits_{q_-}^{q_+}\left[\gamma+q+\varepsilon\varkappa\left(2\gamma q+\frac{3}{2}q^2\right)\right]\frac{\alpha dq}{u_s}=i_c. \tag{2.19}$$

Here, $\gamma=\gamma(l)$ and $\alpha=\alpha(l)$. The substitution of variables

$$(q,\ \lambda)\to(\tau,\lambda);\qquad (dq/d\tau)_\lambda=u_s,$$

$$\text{and } \tau=q=0 \tag{2.20}$$

reduce Eqs. (2.16) and (2.18) to one:

$$d^2q/d\tau^2+q+\gamma+\varepsilon\varkappa(\gamma^2+3\gamma q+{}^3/_2q^2)=dU/dq;$$

$$dU/dq=\alpha\tau+\gamma+\varepsilon\varkappa(\alpha\tau q+\gamma q+\xi(l)+\gamma^2). \tag{2.21}$$

The solution of (2.21) can be represented as

$$q=\alpha\tau+\beta s_\tau-\varepsilon\varkappa z;\quad u_s=\alpha+\beta c_\tau-\varepsilon\varkappa(dz/d\tau);$$

$$z\equiv-\xi(1-c_\tau)+\alpha^2(c_\tau-1+{}^1/_2\tau^2)+$$
$$+{}^1/_2\alpha\beta\tau(s_\tau-\tau c_\tau)+{}^1/_2\beta^2(1-c_\tau)^2+ \tag{2.22}$$
$$+2\gamma\alpha(\tau-s_\tau)+\gamma\beta(s_\tau-\tau c_\tau),\quad \beta=\beta(l).$$

Here and below, we shall use the symbols

$$s_\tau\equiv\sin\tau,\ c_\tau\equiv\cos\tau,\ t_\tau\equiv\operatorname{tg}\tau,$$

$$s_\theta\equiv\sin\theta,\ \text{and}\ c_\theta\equiv\cos\theta.$$

If the axial line is made "symmetric" with respect to τ,

$$q(\tau=\pm\theta)=q_\pm;\quad \theta=\theta(l)\quad(-\theta<\tau<\theta). \tag{2.23}$$

The imposition of conditions (2.20) gives for α, β, γ, ξ, and θ, in the zeroth approximation,

$$q_\pm^0=\pm b_0,\quad \beta_0c_\theta=-\alpha_0,\quad b=\alpha_0\theta+\beta_0s_\theta$$
$$\alpha_0^2=\frac{\pi}{4}\left[\theta\left(1+\frac{1}{2c_\theta^2}\right)-\frac{3}{2}t_\theta\right]^{-1},\quad \gamma_0=\frac{i_c}{2\alpha_0\theta}, \tag{2.24}$$

in first approximation

$$\alpha\equiv\alpha_0+({}^1/_2\varepsilon\varkappa i_c)\alpha_1,$$

$$\beta\equiv\beta_0+({}^1/_2\varepsilon\varkappa i_c)\beta_1,$$

$$\gamma\equiv\gamma_0+\varepsilon\varkappa\gamma_1,$$

$$\beta_1(2\theta c_\theta+\theta/c_\theta-3s_\theta)=$$
$$=(2/\theta)(-{}^3/_4\theta+3s_\theta-2\theta c_\theta+$$
$$+{}^3/_4t_\theta-\theta^2t_\theta+{}^3/_4\theta t_\theta^2-\theta/c_\theta),$$

$$\alpha_1=-\beta_1c_\theta+(1/\theta)\times$$
$$\times[2(1-c_\theta-\theta t_\theta)],$$

$$\theta\gamma_1 = -\tfrac{1}{4}\,(i_c/\alpha_0)^2\,\alpha_1 + \nu,$$

$$\nu \equiv \alpha_0^2\,[\,-\xi\alpha_0^{-2}(\theta - s_\theta) - -\tfrac{1}{3}\,\theta^3 - \tfrac{5}{2}\,\theta + s_\theta + +\tfrac{5}{2}\,t_\theta + \tfrac{1}{2}\,\theta^2 t_\theta - t_\theta/c_\theta],$$

$$\xi s_\theta = \alpha_0^2\,(\tfrac{3}{2}\theta - s_\theta - \tfrac{1}{2}\theta^2 t_\theta - -\tfrac{3}{2}t_\theta + t_\theta/c_\theta)\,, \qquad (2.25)$$

and for positions of the boundaries

$$q_+ \equiv b_0 + \varepsilon\varkappa(\tfrac{1}{2}i_c b_1 + \Delta b)\,,$$

$$-q_- \equiv b_0 + \varepsilon\varkappa(\tfrac{1}{2}\,i_c b_1 - \Delta b)\,,$$

$$b_1 = \alpha_1\theta + \beta_1 s_\theta - -(1/\theta)\,[2\,(\theta - s_\theta) - t_\theta + \theta]\,,$$

$$\Delta b = \alpha_0^2\,[\xi\alpha_0^{-2}\,(1 - c_\theta) - -\theta^2 + \tfrac{1}{2} + \tfrac{1}{2}\,\theta t_\theta + 1/c_\theta - -\tfrac{1}{2}c_\theta^{-2} - c_\theta]\,. \qquad (2.26)$$

Figures 4 and 5 give graphs of the coefficients (2.24)–(2.26) as functions of the parameter θ, which varies in the range $\tfrac{1}{2}\,\pi \le \theta \le \pi$, since by convention the beam is described by one wave Φ.

It is easy to see that $2\theta/\Omega$ is the oscillation time of an electron between the beam boundaries, so that $\pi\Omega/\theta$ is the angular oscillation frequency. This value varies from Ω (zero charge) to $1/2\ \Omega$ (full space charge). The space charge changes the energy of transverse motion $(\alpha_0 + \beta_0)^2\Omega^2 h^2$ and the radius of the electron orbit $b_0 h$. The latter value varies from h (zero charge) to $(\pi/\sqrt{6})h \approx 1$ (3h full charge). The space-charge den-

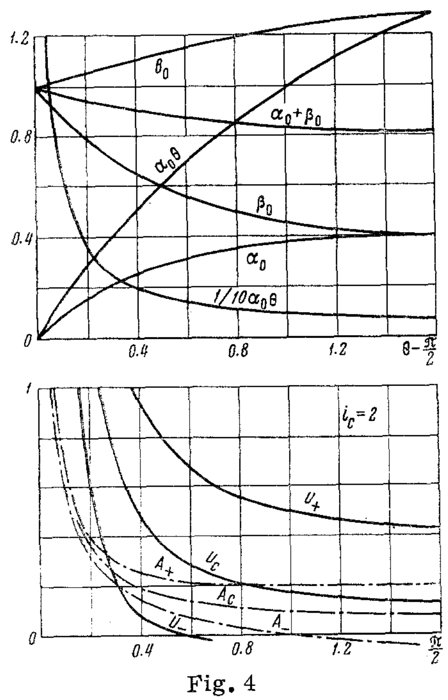

Fig. 4

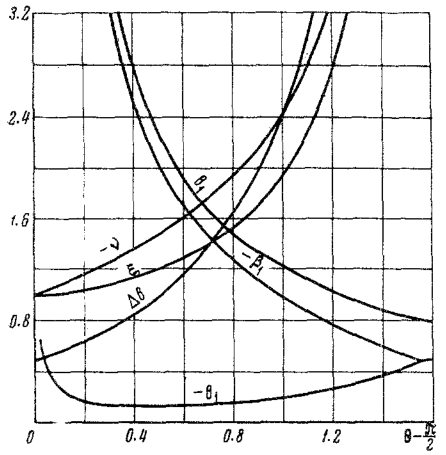

Fig. 5

sity varies in inverse proportion to the field strength on the axis $A_c \sim \gamma_0$ and drops as the strength of the external field near the beam increases. The appearance of singularities in ν, ξ, and Δb at $\theta = \pi$ is due, generally speaking, to improper choice of the axis position.* In fact, the singularity in γ_1 and ξ near $\theta = \pi$ in the formulas for q, u_s, etc., is smoothed by the factors with which γ_1 and ξ enter these formulas. The singularity of Δb, as can be seen from (2.26), causes the selected (symmetric relative to τ) axis to deviate rapidly from the center of the beam when $\theta \to \pi$. The applicability of formulas (2.23)–(2.26) is thereby limited. The calculations for a symmetric position of the axis $q_\pm = \pm$ b with respect to s are not complicated, but the resulting formulas are considerably more cumbersome than (2.25) and (2.26).

Joining the internal field U to the external $V_\pm + B_\pm q + \ldots$ in zeroth approximation gives

$$B^0_\pm = \gamma_0 + \alpha_0\theta;\ 2V_\pm = \gamma_0^2 + b_0^2 \mp 2\ \alpha_0\theta b_0. \qquad (2.27)$$

These relations are shown in Fig. 6 for various beam currents i_c. Figure 4 shows graphs of the dimensionless potentials U and internal-field strength A at the beam boundaries (+, –) and at the center (c), in zeroth approximation.

$$U_\pm = (b_0 \pm \gamma_0)^2;\ A_\pm = \gamma_0 \pm \alpha_0\theta;$$
$$U_c = \gamma_0^2 + b_0^2;\ A_c = \gamma_0\,, \qquad (2.28)$$

2.3°. The second approximation in ε is already a function of the gradients along the longitudinal coordinate $l \equiv \lambda$L. This approximation has such terms as

$$V_s^0 = -(q^0 + \gamma_0)\,u_l^0/u_s^0;$$

$$u_l^0 = (\alpha_0\alpha_0' + \beta_0\beta_0')\,\tau + \alpha_0\beta_0 s_\tau + \beta_0\alpha_0'\,(2s_\tau - \tau c_\tau)\,;$$

$$P \equiv \int V_s^0\,dq =$$

$$= -(\alpha_0\alpha_0' + \beta_0\beta_0')\,[\tfrac{1}{2}\gamma_0\tau^2 + \tfrac{1}{3}\alpha_0\tau^3 + \beta_0\,(s_\tau - \tau c_\tau)] +$$

*The singularity of α_1 and β_1 at the point $\theta = \pi/2$ is not substantial, since α_1 and β_1 participate in the formulas as a sum, which smooths this singularity.

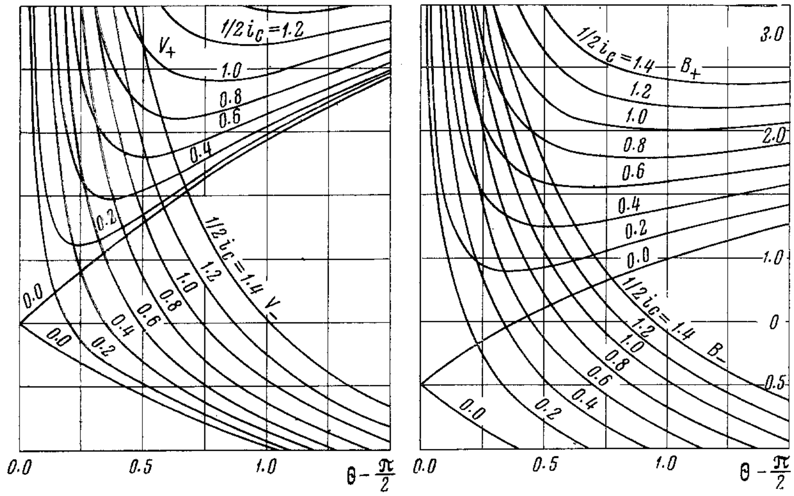

Fig. 6

$$+(\alpha_0\beta_0^{\cdot}-\beta_0\alpha_0^{\cdot})\,[\gamma_0(1-c_\tau)+$$
$$+\alpha_0(1/2\tau^2+s_\tau-\tau c_\tau)+1/2\beta_0(\tau-s_\tau c_\tau)]+$$
$$+\beta_0\alpha_0^{\cdot}\,[\gamma_0(2-2c_\tau-\tau s_\tau)+$$
$$+\alpha_0(3s_\tau-3\tau c_\tau-\tau^2 s_\tau)-1/2\beta_0(3/2\tau-3/2 s_\tau c_\tau-\tau s_\tau^2)]\,;$$
$$\sigma_0\equiv\frac{4\pi\eta}{\Omega^2}\delta_0=-V_s{}^0\frac{\alpha_0}{(u_s{}^0)^2}-\frac{1}{u_s{}^0}\left(\frac{\partial i}{\partial\lambda}\right)_q\,;\qquad \beta^{\cdot}\equiv\frac{d\beta}{d\lambda}\,.$$

Here, as before, the zero subscript denotes zeroth approximation. If, for simplicity, we take a beam without curvature ($k = 0$), for the second approximation we obtain

$$\partial^2 q/\partial\tau^2+\gamma+q+\varepsilon^2\,[(\partial P/\partial\lambda)_q+$$
$$+(\gamma_0+q_0)V_s{}^0+V_s{}^0\;\partial V_s{}^0/\partial q]=\partial U/\partial q\,;$$
$$\partial U/\partial q=\alpha\tau+\gamma-\varepsilon^2\,[\textstyle\int n_2 dq+\int(\partial^2 U_0/\partial\lambda^2)dq-\xi(l)]\,;$$
$$n_2 V_s{}^0=-V_s{}^0\sigma_0-\frac{\partial}{\partial\lambda}\int_{q_-}^{q_-{}^0}\left(\sigma_0 V_l{}^0+\alpha_0\frac{u_l{}^0}{u_s{}^0}\right)dq^0;\qquad \frac{\partial q^0}{\partial\tau}=u_s{}^0\,.$$

2.4°. Relativistic extension of (2.6)–(2.10) results only in a change in the expressions for Φ, H, and Ω:

$$2\eta\Phi\equiv B^2-\varepsilon^2\Gamma_s{}^2-1/g\Gamma_l{}^2-c^2;\;\; B\equiv c+(\eta/c)\varphi\,;$$
$$dH/ds=(4\pi/c\sqrt{g})\;[\rho\Gamma_l+\varepsilon_*{}^2\delta\;(dw/dl)]\,;$$
$$\Omega\equiv(\eta/c)\;H. \tag{2.29}$$

Instead of (2.10), we obtain Eq. (1.17) for B. The remaining equations and conditions have the same form. In zeroth approximation we have

$$\frac{1}{B}\frac{\partial^2 B}{\partial s^2}=\frac{1}{\Gamma_l}\frac{\partial^2\Gamma_l}{\partial s^2}=\frac{4\pi I}{c^3 u_s}\,;\quad u_s{}^2=B^2-\Gamma_l{}^2-c^2. \tag{2.30}$$

This system was studied in [7], but conditions (2.12) and (2.15) were not imposed there. Using the symbols of [7], they can be written as

$$\lambda s_\pm=\pm 2\,[\psi_3(\psi_3-\psi_1)^{-1/2}F(\varphi,n)-$$
$$-(\psi_3-\psi_1)^{1/2}E(\varphi,n)]_{\varphi_m}^{1/2\pi}\,;$$
$$4(\psi_3-\psi_1)^{3/2}\Big[\left(\frac{\psi_3}{\psi_3-\psi_1}\right)^2F(\varphi,n)-$$
$$-\frac{2\psi_3}{\psi_3-\psi_1}E(\varphi,n)-\frac{n^2}{3}\sqrt{1-n^2\sin^2\varphi}\,\times$$
$$\times\sin\varphi\cos\varphi+\frac{2}{3}\frac{2\psi_3-\psi_2+\psi_1}{\psi_3-\psi_1}E(\varphi,n)-$$
$$-\frac{\psi_3-\psi_2}{3(\psi_3-\psi_1)}F(\varphi,n)\Big]_{\varphi_m}^{1/2\pi}=\frac{\pi}{2c}\Omega_* h^2\lambda_*;$$
$$\Omega_*\equiv\frac{\eta}{c}H\,|_{s=s_-}=\mathrm{const}\int_{\psi_1}^{\psi_2}\sqrt{1+\psi^2}\,\mathrm{sh}\,V\frac{d\psi}{\Delta}=\frac{4\pi\eta J_c}{c^3\lambda_*}\,;$$
$$V\equiv\gamma+d_*\int_{\psi}^{\psi_2}\frac{\psi d\psi}{(1+\psi^2)\Delta}$$
$$\varphi_m\equiv\frac{1}{2}\arccos\left(\frac{\psi_2+\psi_1}{\psi_2-\psi_1}\right)^{1/2};$$
$$n^2\equiv\frac{\psi_2-\psi_1}{\psi_3-\psi_1};\quad \lambda_*{}^2\equiv\frac{8\pi\eta}{c^3}\;I\,.$$

Here, F and E are elliptic integrals of the first and second kind; Ω_*, h, and J_c are constants; λ_*, s_*, γ, d_*, and c_* are functions of l; and $\psi_1 < \psi_2 < \psi_3$ are roots of the equation

$$\psi^3+\psi-(1/s_*)\psi^2+c_*=0;\;\text{where}\quad \psi_1<0;$$
$$\Delta\equiv\sqrt{(\psi-\psi_1)(\psi_2-\psi)(\psi_3-\psi)}\,.$$

The solution is represented as $B = c\sqrt{1+\psi^2}\,\mathrm{ch}\,V$; $\Gamma_l = c\sqrt{1+\psi^2}\,\mathrm{sh}\,V$.

As earlier, the given zeroth approximation differs from the exact solution of the corresponding one-dimensional problem in that some of the constants in the latter are interpreted as functions of l, which are related by expressions that follow from (2.12) and (2.15).

§3. **Beam in specified field.** Let a beam drift in a given field with a characteristic inhomogeneity dimension L, so that near the beam axis the field can be represented by a series in s.

3.1°. For a harmonic field we can write

$$\varphi=\psi+As+1/2\,\varepsilon_* kAs^2-1/2\,\varepsilon_*{}^2(Qs^3+\psi'' s^2);$$
$$Q\equiv 1/3\,(A''-2k^2A). \tag{3.1}$$

where $\psi(l)$ is the potential, $A(l)$ is the normal component of the field strength on the axis, and $\psi' \equiv d\psi/dl$. Limiting the accuracy to ε^3, it is not difficult to obtain, taking condition (2.12) into account, the following solution of Eqs. (2.6) and (2.7) for a symmetric position of the axis $s_\pm = \pm a$:

$$2\eta\Phi = u_0^2(1 + \varepsilon_* F_1 + \varepsilon_*^2 F_2);$$
$$u_s = u_0[1 + {}^1/_2\, \varepsilon_* F + {}^1/_2\, \varepsilon_*^2(F_2 - {}^1/_4\, F_1)];$$
$$u_s \equiv \partial w/\partial s,\ u_l \equiv \partial w/\partial l;\ u_0 \equiv \Omega\sqrt{a^2 - s^2};$$
$$\Omega^2 F_1 \equiv k(s\Omega^2 + 3\Gamma\Omega - \eta\ A);$$
$$\Omega^2 F_2 \equiv \eta(Qs + \psi'') + {}^5/_4\, k^2\Omega^2(a^2 + s^2) +$$
$$4k^2 s\Gamma\Omega + 3\ k^2\Gamma^2; \qquad (3.2)$$

$$2\eta\psi = \Gamma^2 + \Omega^2 a^2 + \varepsilon_* k a^2 (3\Gamma\Omega - \eta A) +$$
$$\varepsilon_*^2[\eta\psi'' a^2 + {}^5/_4 k^2 a^4 + 3(ka\Gamma)^2];$$
$$2\eta\ A = \Omega\Gamma\ + \varepsilon_* k(\Omega^2 a^2 + 2\Gamma^2) +$$
$$\varepsilon_*^2\ (\eta Q a^2 + 4\ k^2 a^2\ \Omega\Gamma)\ ; \qquad (3.3)$$

$$a^2\{\Omega^2 + {}^1/_2\ \varepsilon_* k(3\Gamma\Omega - \eta A) +$$
$${}^1/_2\ \varepsilon_*^2[\eta\psi'' + {}^3/_2\ (ka\Omega)^2 + 3\ k^2\Gamma^2 -$$
$$-{}^1/_4 k^2\ (3\ \Gamma\Omega - \eta A)^2]\} = h^2\Omega^2\ . \qquad (3.4)$$

Relations (3.3) and (3.4) can be represented as

$$a_0 = h,\quad \Gamma_0\Omega = \eta A_0;$$
$$2\eta\psi_0 = (\eta/\Omega)^2 A_0^2 + \Omega^2 h^2; \qquad (3.5)$$
$$a_1 = -{}^1/_2 k_0 h\ (\Gamma_0/\Omega);\ \eta\psi_1 = (\Gamma_0/\Omega)\ (\eta A_1 - k_0\ \Gamma_0^2);$$
$$2\Gamma_1\Omega = 2\eta\ A_1 - k_0(h^2\ \Omega^2 + \Gamma_0^2); \qquad (3.6)$$
$$a_2 = -{}^1/_2\ h(k_0\eta\ A_1 + k_1\Gamma_0\Omega +$$
$${}^1/_2\ \eta\psi_0'' - {}^5/_4\ k_0^2\Gamma_0^2)\ \Omega^{-2};$$
$$2\Gamma_2\Omega = 2\eta A_2 - 4\eta A_1 k_0(\Gamma_0/\Omega)$$
$$k_1(h^2\Omega^2 + \Gamma_0^2) + 4k_0^2\Gamma_0^2/\Omega -$$
$$- h^2\ \eta\ Q_0 - k_0^2 h^2\ \Gamma_0\ \Omega,\ldots;$$
$$a \equiv a_0 + \varepsilon_* a_1 + \varepsilon_*^2 a_2;$$
$$\Gamma \equiv \Gamma_0 + \varepsilon_*\Gamma_1 + \varepsilon_*^2\Gamma_2,\ldots\ . \qquad (3.7)$$

Tha half-width of the beam (radius of electron orbit) varies in first approximation as the curvature of the axis and the field strength.* But the axis must be found from the equation $\psi(A, k) = \varphi$, so that A_1, A_2, k_1, and k_2 are corrections to k_0 and A_0.

The latter are determined by the position of the axis $y(x)$ in zeroth approximation,

$$2\eta\varphi = (\eta/\Omega)^2 A^2 + \Omega^2 h^2;$$
$$A \equiv [\varphi_x dy/dx - \varphi_y]\,[1 + (dy/dx)^2]^{-1/2}. \qquad (3.8)$$

If we add the constant $\Omega^2 h^2$ to the potential φ in the equation for the electron trajectory (1.14) and expand the right side in powers of $\varepsilon_* \sim k$, we obtain

$$2\eta\varphi = \Omega^2 h^2 + (\eta/\Omega)^2 A^2\ (1 - 2\varepsilon_*\, k\,\eta\Omega^{-2}\ A + \ldots);$$
$$\varphi_x \equiv d\varphi/dx. \qquad (3.9)$$

which coincides in first approximation with the equation for the beam axis which follows from (3.5) and (3.6). With accuracy to ε^2, therefore, the beam axis coincides with a properly produced electron trajectory.

For the velocity Γ_l and density Eqs. (2.7)–(2.9) give

$$\Gamma_l = \Omega s + \gamma - {}^1/_2\ \varepsilon_*\ \Omega k s^2;\ \Gamma_s \sim \varepsilon_*$$
$$\delta = I'/\Omega;$$
$$(1 - \varepsilon_* k s)\ \rho u_s = I + {}^1/_2\ \varepsilon_*^2 I''\ \ (a^2 - s^2)\,. \qquad (3.10)$$

In view of (3.2) and (3.10), condition (2.15) yields

$$\pi I\ \{\Gamma\Omega^2 + {}^1/_2\varepsilon_*\ k(\Omega^3 a^2 - 3\Gamma^2\Omega + \eta A\Gamma) -$$
$$-\ \varepsilon_*^2\Gamma[{}^3/_4\,(ka\Omega)^2 + {}^1/_2\eta\psi'' +$$
$$+ {}^1/_4\ a^2(Q/A)\ \Omega^2]\} + {}^1/_4\ \varepsilon_*^2\ \pi\ I''\ a^2\Gamma = J_c\Omega^3\,.$$

Hence, and from (3.5)–(3.7) and (3.10), it follows, with accuracy to ε_*^2, that

$$\rho = J_c\Omega\ \frac{1 + \varepsilon_* k\,({}^1/_2 s + \eta A\Omega^{-2})}{\pi\eta A\ \sqrt{\Omega^2 - s^2}}\,;$$
$$\int_{-a}^{a} \rho ds = \frac{\Omega J_c}{\eta A}\ (1 + \varepsilon_* k\eta A\Omega^{-2})\,. \qquad (3.11)$$

It is appreent that a drift due to centrifugal force with velocity $\sim k(\eta A)^2\,\Omega^{-3}$ is added to the main drift, which has velocity $(\eta/\Omega)A$. The obtained velocity and density distributions are functions of the parameter η, which can vary over a finite range, according to the geometry of the emitter, so that the full beam has a multivalued velocity field and is represented as the aggregate of the double-flow beams considered above. The latter are conveniently called tubes of flow. A tube of flow is the set of trajectories on the interval dh. This set, generally speaking, does not completely fill the volume of the tube. The density ρ, therefore, can be interpreted as the density of the charge, which spread from the true trajectcries over the possible adjacent trajectories in the tube to a continuous distribution.

3.2°. It is easy to show that in zeroth approximation, according to (3.8), the axis oscillates as a quasi-cycloid between the two discriminant curves

$$2\eta\varphi = \Omega^2 h^2;$$
$$2\eta\varphi = \Omega^2 h^2 + (\eta/\Omega)^2\ (\varphi_x^2 + \varphi_y^2)\ ;\ \varphi_x = \partial\varphi\ /\ \partial x \qquad (3.12)$$

These curves, bounding, generally speaking, a very narrow band [4], intersect and branch at the singular point of the field $\varphi_x = \varphi_y = 0$. Therefore, the beam axis also undergoes branching at this point, i.e., at the singular points $\varphi_x = \varphi_y = 0$, the tubes of flow branch.

*The orbit expands when the centrifugal force and electric field act on the electron in one direction; otherwise, it contracts.

Moreover, as is easy to see from (1.8), the equation of the axis of a dense beam in zeroth approximation coincides in form with (3.8). Therefore, splitting of a dense beam is possible at the singular point $\varphi_x = \varphi_y = 0$.

3.3°. For a weak field

$$\varphi = \psi(l) + \varepsilon_* sA(l) + \varepsilon_*^2 s^2 B(l) + \varepsilon_*^2 c(l) s^3 + \ldots \tag{3.13}$$

similar calculations give

$$\Gamma_l = \Omega s + \varepsilon_* (\Gamma - {}^1/_2\, \Omega\, ks^2) + \varepsilon^3 R;$$

$$R \equiv -{}^1/_{24}\, \Omega\, k''\, (a^2 - s^2)^2;$$

$$u_s = u_0\, [1 + {}^1/_2\, \varepsilon_* ks + {}^1/_2\, \varepsilon_*^2 (F_2 - {}^1/_4\, k^2 s^2) +$$

$$+ {}^1/_2\, \varepsilon_*^3\, s\, (F_3 - {}^1/_2\, kF_2 + {}^1/_8\, k^3 s^3)];$$

$$F_2 \Omega^2 \equiv -2\eta B + {}^5/_2\, k^2 \Omega^2 (a^2 + s^2) + 3k\, \Gamma\Omega;$$

$$u_0 \equiv \Omega \sqrt{a^2 - s^2};$$

$$F_3 \Omega^2 \equiv -2\eta c + {}^3/_2 k^3 \Omega^2 (a^2 + s^2) + 4k^2 \Gamma\Omega - 2R\Omega\, (a^2 - s^2)^{-1};$$

$$u_l = -{}^1/_2\, \varepsilon_* u_0\, \{{}^1/_3\, k'\, (a^2 - s^2) +$$

$$+ {}^1/_2\, \varepsilon_* [2\eta B \Omega^{-2} - {}^5/_2\, kk' a^2 - ({}^3/\Omega)\, (k\Gamma)'] s +$$

$$+ {}^1/_4\, \varepsilon_*\, kk' s\, (a^2 - 2s^2)\};$$

$$\Gamma_s = {}^1/_2 \varepsilon_*\, \{{}^1/_3\, k'\, (a^2 - s^2)\, [\Omega s + \varepsilon_* (\Gamma + 2\, k\Omega s^2) +$$

$$+ {}^1/_4\, \varepsilon_*\, kk'\, \Omega s^2\, (a^2 - 2s^2)] +$$

$$+ {}^1/_2\, \varepsilon_* \Omega s^2\, [2\eta B \Omega^{-2} - {}^5/_2\, kk' a^2 - ({}^3/\Omega)\, (k\Gamma')]\}. \tag{3.14}$$

For the parameters of the tube of flow a, Γ, and Ψ we obtain

$$a \equiv a_0 + \varepsilon_*^2 a_2;\ \Gamma \equiv \Gamma_0 + \varepsilon^2 \Gamma_2;\ \psi = \psi_0 + \varepsilon^2 \psi_2; \tag{3.15}$$

$$a_0 = h;\ \Gamma_0 \Omega = \eta A_0 - {}^1/_2\, k_0 \Omega^2 h^2;\ \ 2\eta\psi_0 = \Omega^2 h^2;$$

$$a_2 = {}^1/_4\, \eta h \Omega^{-2}\, (2B_0 - 3k_0 A_0);$$

$$2\eta\psi_2 = (\eta A_0/\Omega)^2 + {}^1/_2\, \eta h^2\, (k_0 A_0 - 2B_0);$$

$$2\Omega\Gamma_2 = 2\eta A_2 - k_2 \Omega^2 h^2 + 2\eta c_0 h^2 - 2k_0 (\eta A_0/\Omega)^2 -$$

$$- {}^1/_2\, k_0 \eta\, h^2\, (k_0 A_0 + 2B_0). \tag{3.16}$$

For definiteness, let field φ be harmonic. Then

$$2B = kA - \psi'';\ \ 6c = 2k^2 A - 2k\psi'' - k'\psi' - A''. \tag{3.17}$$

From (2.8), (2.9), (3.14), and condition (2.15) we obtain, for the density,

$$I = \frac{J_c \Omega^2}{\pi \eta A_0} (1 + \varepsilon_*^2 f);$$

$$f = -\frac{A_2}{A_0} + 2k_0 \frac{\eta A_0}{\Omega^2} + \frac{h^2}{3} \left(k_0^3 + \frac{A_0''}{A_0}\right);$$

$$\rho = \frac{J_c \Omega}{(1 - \varepsilon_* ks)\, \pi\eta A_0} \left\{ \frac{1 + \varepsilon_*^2 f}{\sqrt{a^2 - s^2}} + \right.$$

$$\left. + \left[\left(\frac{A_0'}{A_0}\right)^2 - \frac{A_0''}{2A_0}\right] \sqrt{a^2 - s^2} \right\}. \tag{3.18}$$

If it is assumed that φ is a traveling-wave field in a frame of reference that moves uniformly with the wave, then (3.14)-(3.18) are applicable to the problem that was solved in [3] by the averaging method. From (3.16) and (3.17) follows a result (for example, for the axis position) that agrees in second approximation with the line of orbit centers that was obtained in [3],

$$2\eta\varphi = \varepsilon_*^2 (\eta/\Omega)^2 (\nabla\varphi)^2 + \Omega^2 h^2.$$

Similar calculations for a beam in a frame of reference that rotates uniformly with the wave give, for the axis position, formula (23) of [4].

Appendix. If, as is usually done [3,4], it is assumed that (in a frame of reference which rotates with the rotating wave of the fundamental field harmonic for a generating magnetron at angular velocity ω = const) the problem of calculation of an electron process in the magnetron may be formulated as a stationary problem, the described method may be used to calculate the electron "spokes" of the magnetron. These spokes, as is generally recognized, are relatively narrow beams that drift in the rotating system from the cathode (k) to the anode (a) (see Fig. 7).

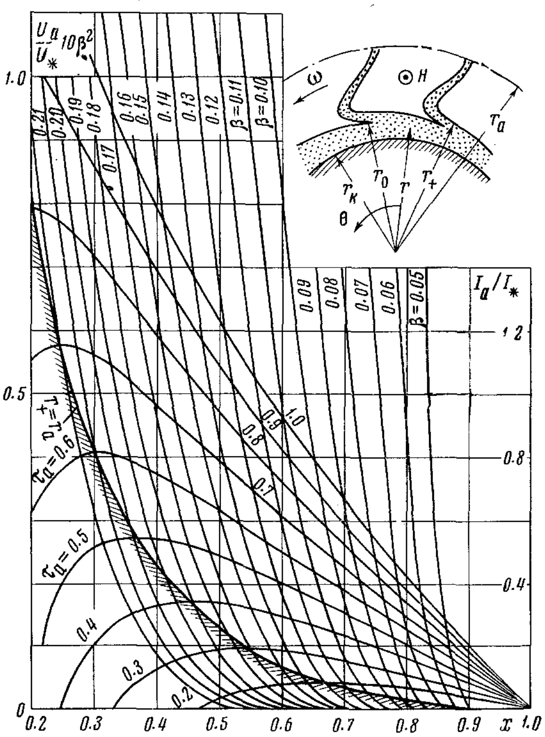

Fig. 7

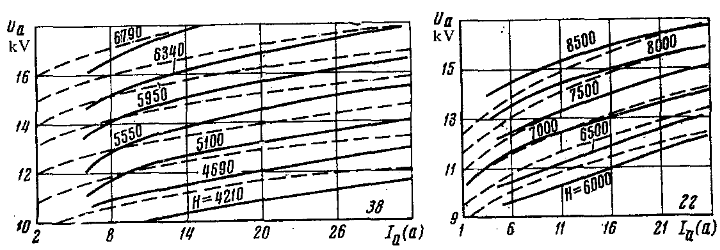

Fig. 8

The following preliminary estimate of the volt-ampere characteristic of a magnetron operating under full space-charge limitation was verified. Only the considerations of dense-beam branching discussed in 3.2° were used.

Since the high-frequency field E decreases rapidly from the anode to the cathode [2–4], the field in the near-cathode region (at the "hub"), ignoring perturbation from the spokes whose charge from the cathode $\sim 1/E$ rapidly decreases can be roughly represented as the field of a one-dimensional (r) cylindrical single-flow beam (hub). The density of this beam ρ, the space-charge field φ, and the azimuthal electron velocity v_θ in a rotating frame of reference have the form

$$\rho = \frac{\Omega^2}{8\pi\eta}\left(1+\frac{r_k^4}{r^4}\right); \quad 2\eta\varphi = \frac{1}{r^2}v_\theta - \Omega^2 r_k^2; \quad \Omega \equiv \frac{\eta}{c}H;$$
$$v_\theta = {}^1/_2\,\Omega\varkappa\,(r^2 - r_k^2); \ \beta \equiv \omega/\Omega; \ \varkappa \equiv 1 - 2\beta.$$

The singular point of the field is obviously found near the line $r = r_0$

$$(\partial\varphi/\partial r)_{r=r_0} = 0, \ r_0 = r_k\varkappa^{-1/2} \qquad (A.1)$$

A spoke with current $2I_a/N$ branches off on this line from the near-cathode beam:

$$\frac{I_a}{I_*} \approx \frac{h_*}{I_*}\int_{r_0}^{r_+}\rho\frac{v_\theta}{r}dr =$$
$$= \frac{1}{40\beta^3}\left\{\varkappa\left(\frac{1}{x} - x - \frac{1}{\varkappa} + \varkappa\right) - \ln\frac{\varkappa}{x} + \frac{x^2}{2} - \frac{\varkappa^2}{2}\right\};$$
$$x \equiv \left(\frac{r_k}{r_+}\right)^2; \quad I_* \equiv \frac{10Nh_*\omega^3 r_k^3}{16\pi\eta}. \qquad (A.2)$$

Here, $N/2$ is the number of spokes, I_a the anode current, and h_* the height of the working part of the cathode. A well-known relation in [8] links the upper effective boundary of the near-cathode beam r_+ with the constant anode-cathode potential difference U_a:

$$10\beta^2\frac{U_a}{U_*} = \frac{1-x}{4x}\left[1 - x + (1+x)(2\tau_a + \ln x)\right];$$
$$\tau_a \equiv \lg\frac{r_a}{r_k}; \quad V_* \equiv \frac{5\Omega^2 r_k^2}{\eta}. \qquad (A.3)$$

In conjunction with (A.2), it gives the desired characteristic $I_a(U_a)$. These relations are shown graphically in Fig. 7. The volt-ampere characteristics of ten types of magnetrons described in [8] were calculated by (A.2) and (A.3). Comparison with the experimental characteristics in [8] showed agreement within 20-5%. For four types (4j50, 3j31, 22-x, and 38-cavity) the voltage deviations were within 10-5%. In Fig. 8, the experimental characteristics are shown by solid lines and the calculated by dashed lines.

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